# The Central Hopf Algebra

Wednesday, September 23, 2020 5:39 PM

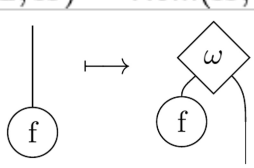
Throughout the talk:

- There is a fixed base field $\boldsymbol{k} = \overline{\boldsymbol{k}}$.
- All Cats are finite braided tensor categories over $\boldsymbol{k}$.
- Assume pivotal/spherical/ribbon when needed.

- Nonsemisimple modular categories. (Lyu. mid '90s)
- Definition uses:
  - $H := \int^{X \in \mathcal{C}} X^* \otimes X$ (An object)
  - $\omega : H \otimes H \to \mathbb{1}$ (A pairing)
  - $Q : H \to H^*$ (A comparison map)

Check out:
**Non-Semisimple Topological Quantum Field Theories for 3-Manifolds with Corners**
by Kerler and Lyubashenko ('01)

- Thm(Shimizu, '16) TFAE:
  - $\omega$ is nondegenerate $\iff$ $Q$ is an iso (Lyubashenko's `*modular*')
  - $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}} \simeq \mathcal{Z}(\mathcal{C})$ (←*factorizable*)
  - $\Omega : \mathrm{Hom}(\mathbb{1}, H) \to \mathrm{Hom}(H, \mathbb{1})$ is an iso




(*weakly factorizable*)

  - The Müger center of C is trivial.

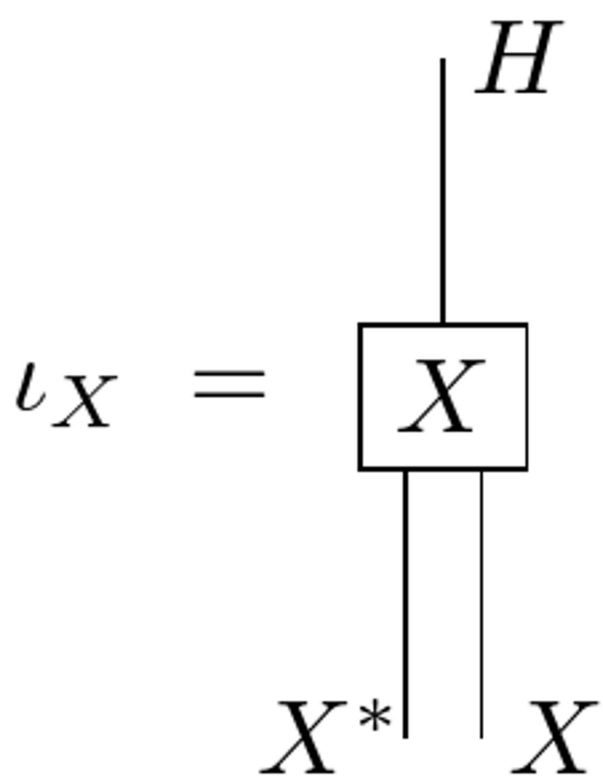
The main technical tool is $H := \int^{X \in \mathcal{C}} X^* \otimes X$

( For semisimple Cats: $H \cong \bigoplus X^* \otimes X$ )

Here's the U-P: ( it's a coend )

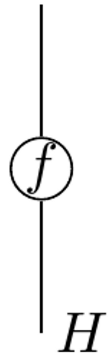
$$
\forall f : A \to B, \quad
\begin{tikzcd}
B^* \otimes A \arrow[r, "id \otimes f"] \arrow[d, "f^* \otimes id"'] & B^* \otimes B \arrow[d, "\iota_B"'] \arrow[ddr, bend left] & \\
A^* \otimes A \arrow[r, "\iota_A"] \arrow[drr, bend right] & \int^{X \in \mathcal{C}} X^* \otimes X \arrow[dr, dotted] & \\
 & & W
\end{tikzcd}
$$

Graphically, we will denote the maps $\iota_X : X^* \otimes X \to H$ by

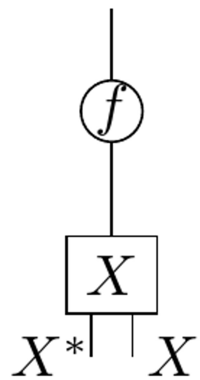



The U-P of $H$ is about ***maps out***, and so…

If you want to know this...



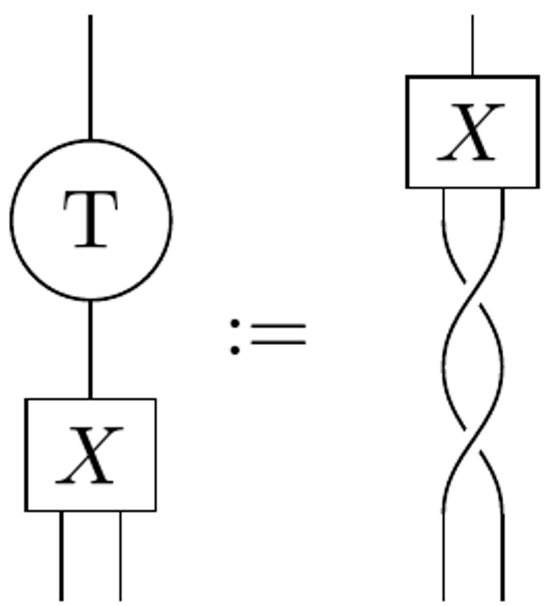

it suffices to check these!




$\forall\, X \in \mathcal{C}$

Uses the philosophy of `***generalized points***'

<u>EG</u>:

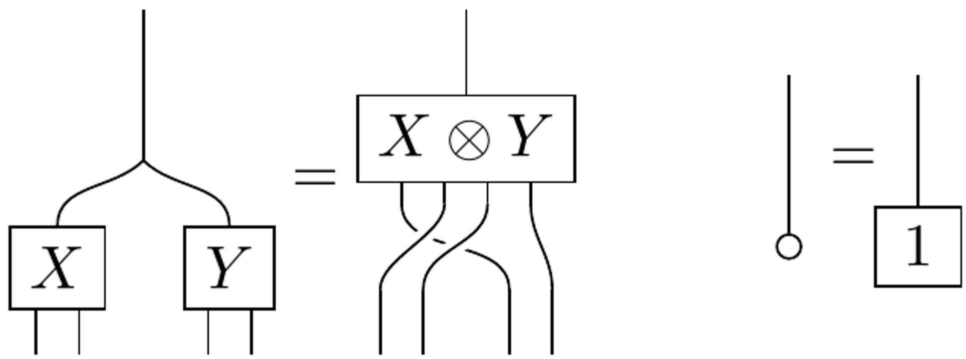


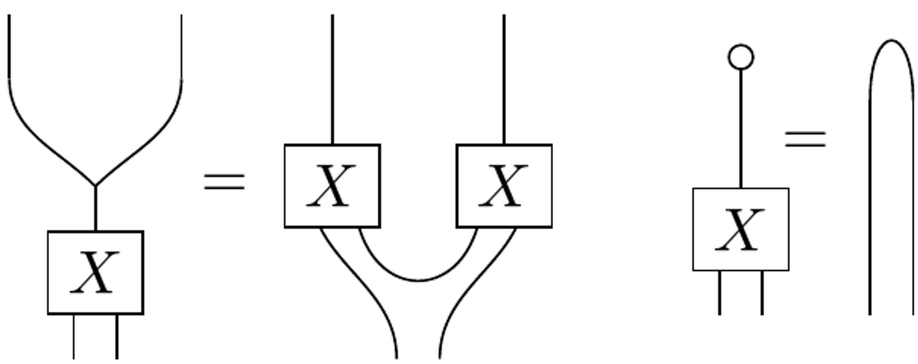

this is uniquely defines a map
$T: H \to H$

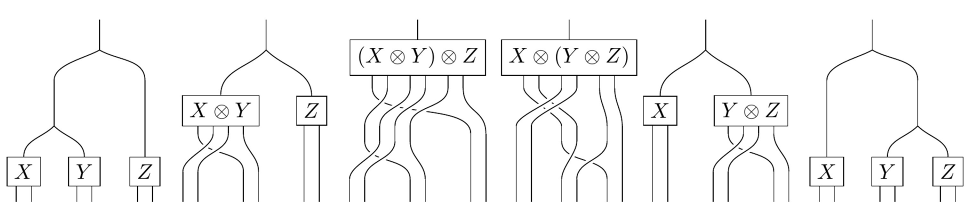
The dual $H^*$ also satisfies a U-P: (this one's an end)

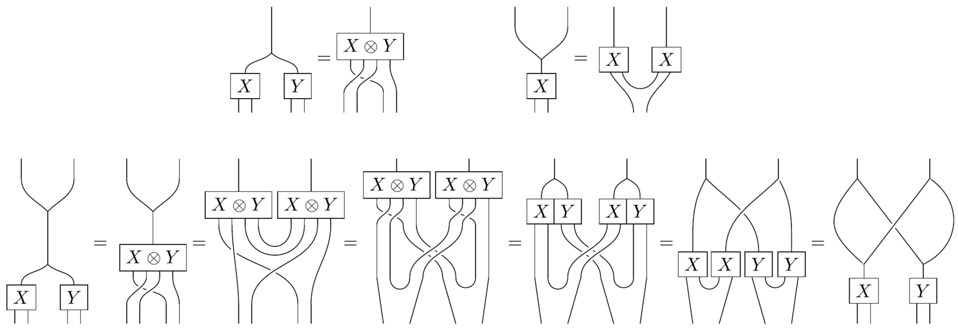
$$
\begin{tikzcd}
 & W \arrow[d, dotted] \arrow[rd, bend left] \arrow[dd, bend right] & \\
\forall f : A \to B, \quad & \int_{X\in\mathcal{C}} X \otimes X^* \arrow[r, "\pi_B"] \arrow[d, "\pi_A"] & B \otimes B^* \arrow[d, "id\otimes f^*"] \\
 & A \otimes A^* \arrow[r, "f\otimes id"] & B \otimes A^*
\end{tikzcd}
$$

As it turns out, $H$ is a Hopf algebra:



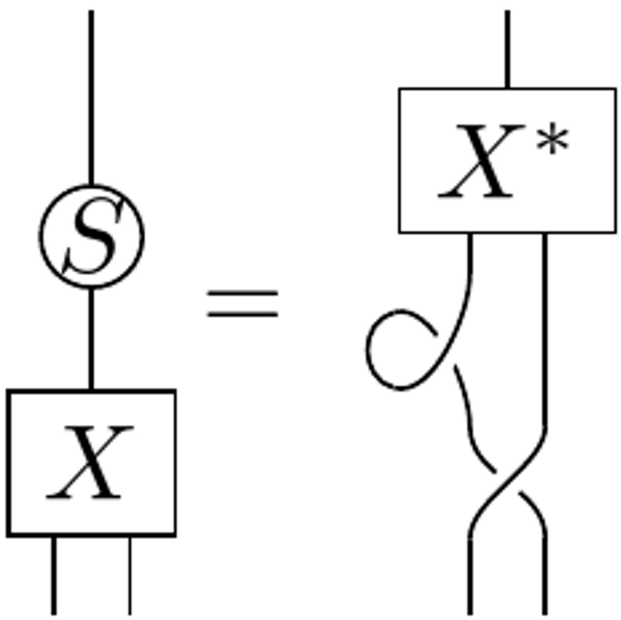



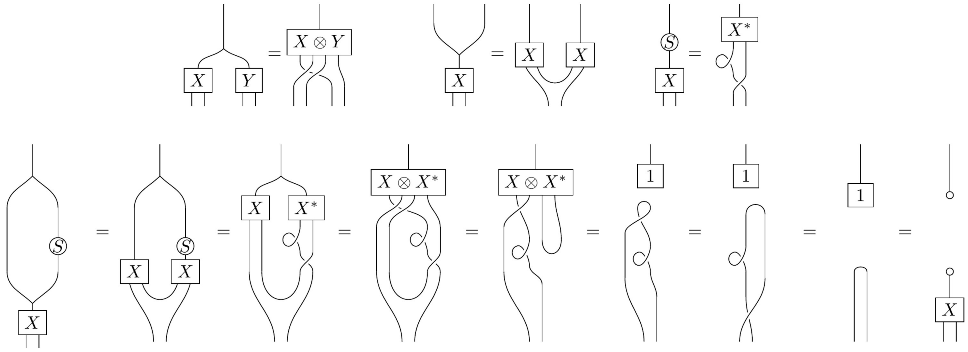






Great!  So this shows that it's a bialgebra…

So what's the Antipode?



Observation: The algebra structure is related to the braiding.

Some (co)end calculations

Fact: $\int_{X \in \mathcal{C}} \mathsf{Hom}(FX, GX) \cong \mathsf{Nat}(F, G)$

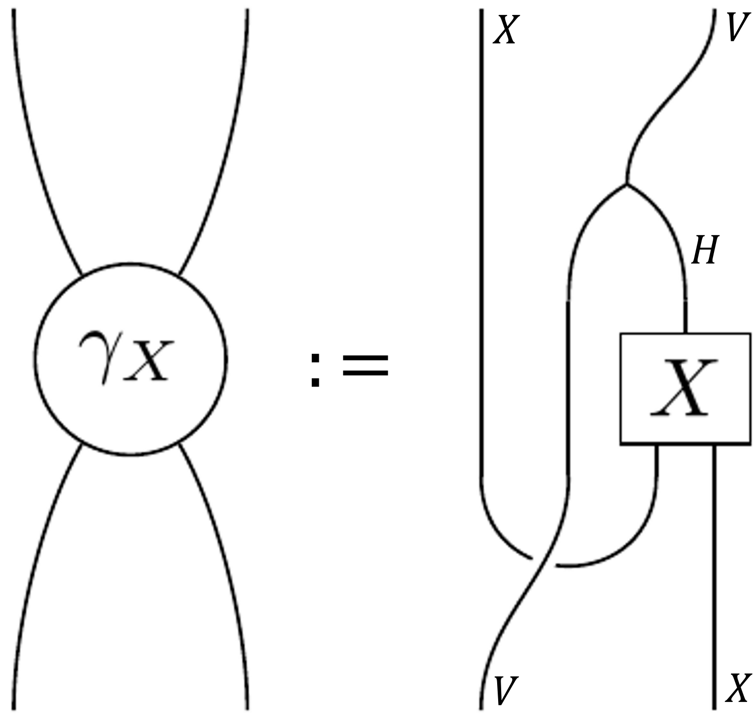
Check out:
**Coend Calculus** by Fosco Loregian (on the arXiv)

Using this fact, we can observe:

$$\begin{aligned}
\mathsf{Hom}(V \otimes H, V) &\cong \mathsf{Hom}\left(V \otimes \int^{X \in \mathcal{C}} X^* \otimes X \,,\, V\right) \\
&\cong \int_{X \in \mathcal{C}} \mathsf{Hom}(V \otimes X^* \otimes X \,,\, V) \\
&\cong \int_{X \in \mathcal{C}} \mathsf{Hom}(X^* \otimes V \otimes X \,,\, V) \\
&\cong \int_{X \in \mathcal{C}} \mathsf{Hom}(V \otimes X \,,\, X \otimes V) \\
&\cong \mathsf{Nat}\big((V \otimes -), (- \otimes V)\big)
\end{aligned}$$

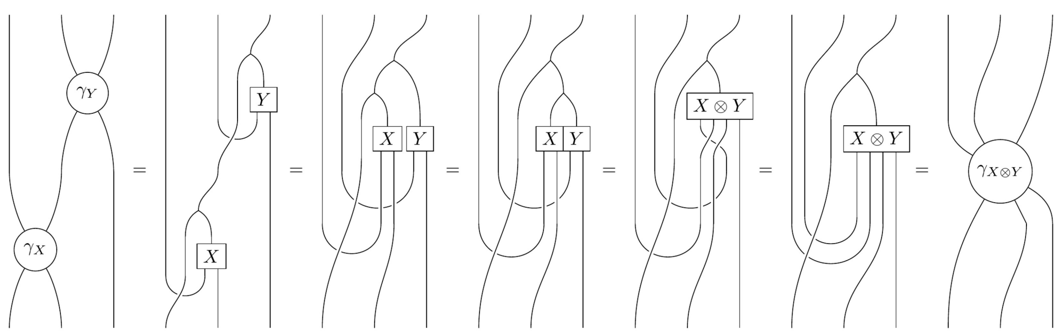
Thus right $H$ modules are candidates for half-braidings!

Compare with

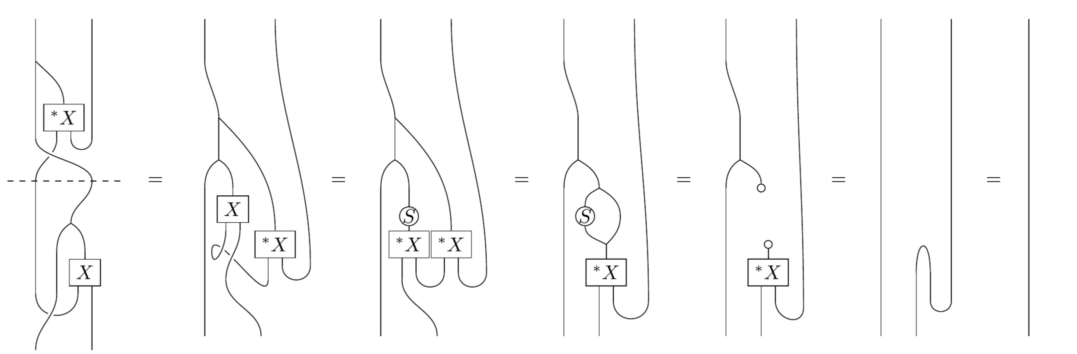



<u>Claim</u>: This works!

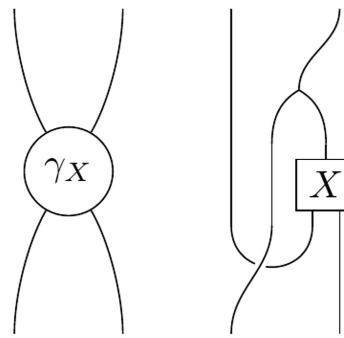
The hexagon: $(id \otimes \gamma_Y) \circ (\gamma_X \otimes id) = \gamma_{X \otimes Y}$

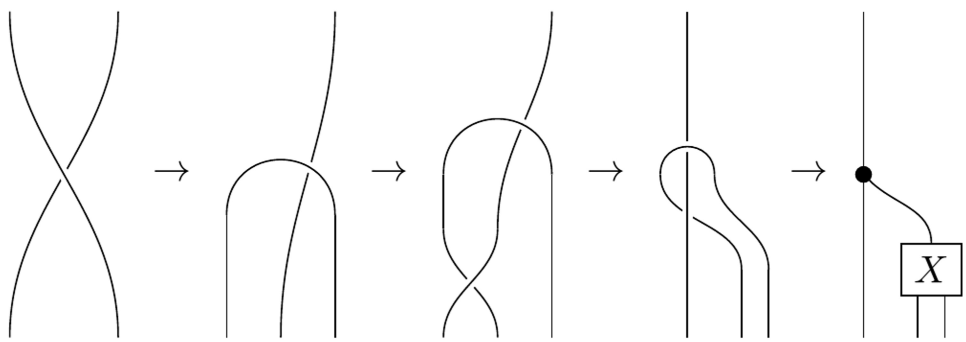



Here's a verification of invertibility:



This construction can be reversed:




<u>Question</u>: What module structure gives the over-braiding?

- Above computations show $C_H \cong Z(C)$.
- ( Sidenote → It can be shown that $C_{H^*} \cong C \boxtimes C^{rev}$ )
- There is an adjunction:

$$\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}\Big((V \otimes H, \gamma), (W, \zeta)\Big) \cong \mathrm{Hom}_{\mathcal{C}}\Big(V, W\Big)$$

- Free modules give a left adjoint, so...
- $H \cong UF(\mathbf{1})$ (←You've probably seen this object before)

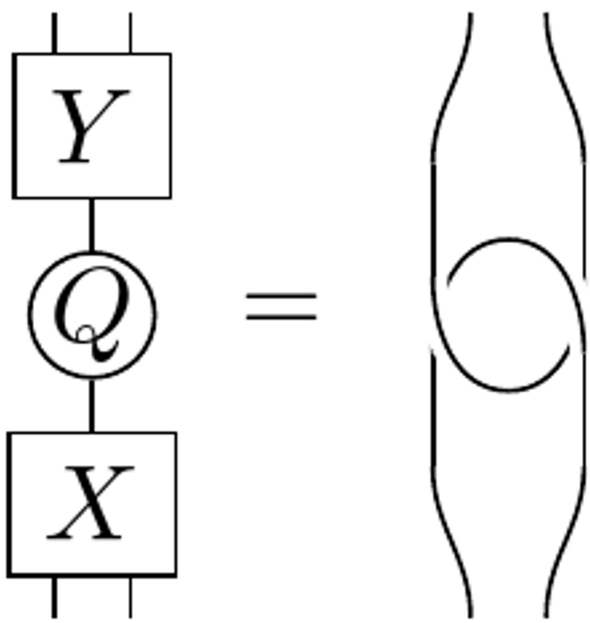
Let's look at one of Shimizu's proofs.

Strategy:

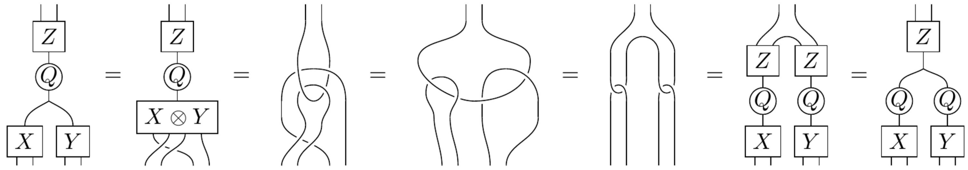
1. Take an algebra map: $\varphi: A \to B$
2. Consider the 'restriction of scalars' fuctor $\varphi^*: C_B \to C_A$
3. $\varphi^*$ is an equivalence $\Leftrightarrow$ $\varphi$ is an isomorphism
4. Use the comparison map $Q: H \to H^*$
5. Profit

$$
\begin{tikzcd}
H \arrow[d, "Q"'] & & \mathcal{C}_H \arrow[rr, "\cong"] & & \mathcal{Z}(\mathcal{C}) \\
H^* & & \mathcal{C}_{H^*} \arrow[u, "Q^*"] & & \mathcal{C} \boxtimes \mathcal{C}^{rev} \arrow[ll, "\cong"] \arrow[u]
\end{tikzcd}
$$

Let's see this $Q$:

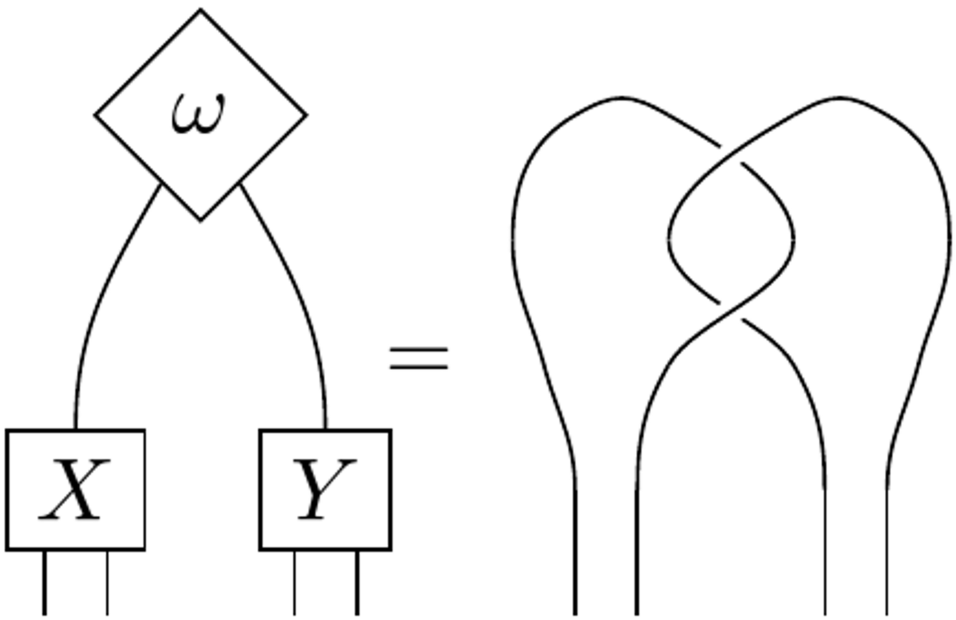



Why is an a morphism of algebras?




It preserves coproducts for analogous reasons.

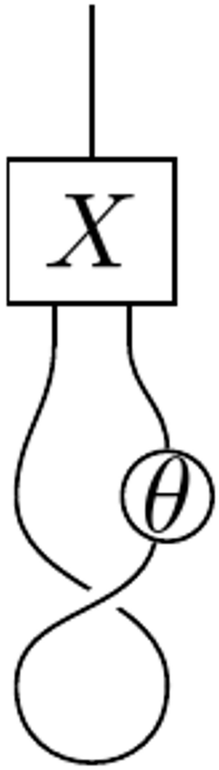
The adjoint of $Q$ is a pairing $\omega: H \otimes H \to \mathbf{1}$:




Let's examine how this relates to the classical interpretation involving the S-matrix:

When $C$ is **semisimple,** there is a basis of $Hom(\mathbf{1}, H)$ indexed by the simples:

$\chi_X :=$

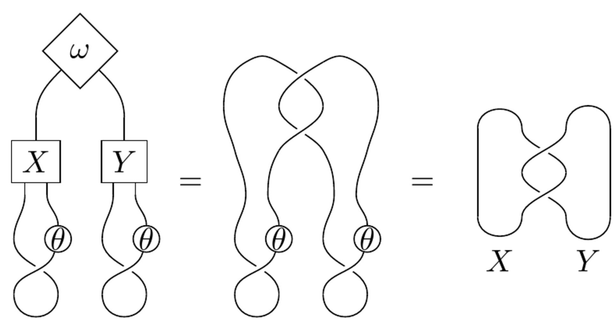



The twist corrects for the braiding

Using this basis, we obtain:

$\Omega_*: Hom(\mathbf{1}, H) \otimes Hom(\mathbf{1}, H) \to Hom(\mathbf{1}\,, \mathbf{1}) \cong \boldsymbol{k}$

$[\Omega_*]_{X,Y} =$




$= [S]_{X,Y}$

Thus S is nondegenerate precisely when this pairing on hom-spaces is nondegenerate. This is the same as saying that the morphism Ω is an isomorphism of vector spaces, *i.e.* that $C$ is *weakly factorizable* in the sense of Takeuchi.